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On the Rate of Convergence to the Normal Law for Solutions of the Burgers Equation with Singular Initial Data

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We study the scaling limit of random fields which are the solutions of a nonlinear partial differential equation known as the Burgers equation, under stochastic initial condition. These are assumed to be a Gaussian process with long-range dependence. We present some results on the rate of convergence to the normal law.

KEY WORDS: Nonlinear waves; scaling limit; Gaussian initial conditions; Hermite expansion; long-range dependence, rate of convergence.

1. INTRODUCTION

Burgers' equation is known to describe various physical phenomena, such as nonlinear waves (see, for example, refs. 3, 28, and 7), the distribution of self-gravitating matter in the universe,[1] and other types of flows. Some models related to the one-dimensional Burgers equation have also been worked out in economics.[8] Rosenblatt first considered the Burgers equation with random initial data.[21,22] Many people have recently investigated solutions of the Burgers equation depending on different types of random initial conditions. In particular, Bulinskii and Molchanov,[2] Giraitis *et al.*,[6] and Albeverio *et al.*[1] studied solutions of the Burgers equation when the initial condition is either a Gaussian random field or a shot-noise random field with weak or strong dependence. Leonenko *et al.*[13,14,16,17] present Gaussian and non-Gaussian limit distributions of solutions of initial-valued problems for the Burgers equation when the initial condition

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is either a Gaussian homogeneous isotropic random field or a chi-square field with long-range dependence. Deriev and Leonenko[4] considered the Gaussian limit field for a scaling limit of solutions of the Burgers equation under suitable non-Gaussian initial conditions with weak dependence. In the Gaussian model with nonintegrable oscillating correlations the limit law of solutions is non-Gaussian.[25] We also mention the results of Surgailis and Woyczynsky[26] on the Burgers equation with nonlocal shot noise data. Sinai[23] and Holden *et al.*[9] considered the nonhomogeneous Burgers equation submitted to initial random conditions, deriving some asymptotic properties of limit solutions when the forcing term displays some periodicity and Hermite expansions for the solutions. Sinai[24] considered the statistics of shocks of the solutions of the Burgers equation. Majda[18] present explicit inertial range renormalization theory in a model for turbulent diffusion with large Reynolds number and long-large correlations of the initial conditions.

Many authors have analyzed processes and fields with long-range dependence (e.g., refs. 5, 27, and 10). Some results on the rate of convergence to the normal law for integral functionals of homogeneous isotropic Gaussian random fields under strong dependence were considered by Leonenko.[11]

In this paper we present results on the rate of convergence to the normal law of the solutions of the Burgers equation with strongly dependent Gaussian initial condition.

2. PRELIMINARIES

We consider the one-dimensional Burgers equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\mu\frac{\partial^2 u}{\partial x^2}, \qquad t>0, \quad x\in\mathbf{R}^1 \tag{2.1}$$

subject to the initial condition

$$u(0,x)=u_0(x)=\frac{d}{dx}v(x), \qquad x\in\mathbf{R}^1 \tag{2.2}$$

which describes the evolution of the velocity field $u(t,x)$, $(t,x)\in[0,\infty)\times\mathbf{R}^1$.

Equation (2.1) is a simplified version of the Navier–Stokes equation with $R=1/\mu$ corresponding to the Reynolds number.

Despite its apparent simple form, the Burgers equation (2.1) encompasses many of the important features of the fluid flow and has furthermore found many applications in other areas.

A crucial property of (2.1) is that it can be linearized by the so-called Hopf–Cole transformation (see also refs. 28, 7, and 2)

$$u(t, x) = -2\mu \frac{\partial}{\partial x} \log z(t, x)$$

This transformation reduces (2.1) to the linear diffusion equation

$$\frac{\partial z}{\partial t} = \mu \frac{\partial^2 z}{\partial x^2}$$

subject to the initial condition

$$z(0, x) = \exp\left\{-\frac{v(x)}{2\mu}\right\}$$

The solution to Eq. (2.1) in the class of potential fields $u(t, x) = (\partial/\partial x)\, v(t, x)$ is given by the explicit Hopf–Cole formula

$$u(t, x) = \frac{\int_{-\infty}^{\infty} [(x-y)/t]\, g(t, x-y) \exp[-v(y)/2\mu]\, dy}{\int_{-\infty}^{\infty} g(t, x-y) \exp[-v(y)/2\mu]\, dy} = \frac{I(x, t)}{J(x, t)} \tag{2.3}$$

where $v(x) = v(0, x)$ is the initial potential [see (2.2)] and

$$g(t, x-y) = (4\pi\mu t)^{-1/2} \exp[-|x-y|^2/(4\mu t)], \qquad x, y \in \mathbf{R}^1, \quad t > 0$$

is the Gaussian (heat) kernel.

Let now $(\Omega, \mathfrak{F}, P)$ be a complete probability space. We assume that the initial potential $v(x) = \xi(\omega, x)$, $\omega \in \Omega$, $x \in \mathbf{R}^1$, is a random process satisfying the following condition:

Condition A. Let $\xi(\omega, x) = \xi(x)$, $\omega \in \Omega$, $x \in \mathbf{R}$, be a real, measurable, mean-square differentiable stationary Gaussian process with $E\xi(x) = 0$, $E\xi^2(x) = 1$, and correlation function

$$B(x) = E\xi(0)\, \xi(x) = B(|x|) = \frac{L(|x|)}{|x|^\alpha}, \qquad 0 < \alpha < 1, \quad x \in \mathbf{R}^1$$

where $L(t)$, $t>0$, is a slowly varying function for large values of t and bounded on each finite interval, i.e., the function L: $(0, \infty) \to (0, \infty)$ such that for all $\lambda > 0$

$$\lim_{t \to \infty} \frac{L(\lambda t)}{L(t)} = 1$$

When the initial potential is random we focus our attention on the statistical properties of the solution (2.3), in particular, its limiting distribution as t tends to infinity. For various forms of $v(x)$, the problem was considered in, e.g., refs. 22, 2, 13, 6, 14, 4, 16, 17, 25, and 26. Here we consider the Gaussian case with long-range dependence, i.e., $v(x) = \xi(x)$, $x \in \mathbf{R}^1$, is a stationary Gaussian process whose covariance decays slowly as $|x| \to \infty$ (or, equivalently, the spectral density is singular).

Let $u = u(t, x)$, $(t, x) \in [0, \infty) \times \mathbf{R}^1$, be the solution of the Cauchy problem (2.1), (2.2) with the random initial condition satisfying Condition A. The main result of this paper concerns the limiting behavior of the process $u = u(t, a\sqrt{t})$, $a \in \mathbf{R}^1$, when $t \to \infty$.

Let

$$\varphi(w) = \frac{1}{(2\pi)^{1/2}} \exp\left(-\frac{w^2}{2}\right), \qquad w \in \mathbf{R}$$

be the density function of the Gaussian random variable with parameters (0, 1),

$$\Phi(z) = \int_{-\infty}^{z} \varphi(w)\, dw \tag{2.4}$$

An application of the ideas and methods of Dobrushin and Major,[5] Taqqu,[27] and Leonenko and Olenko[12] yields the following theorem proved in refs. 13 and 14.

Theorem 2.1. Let $u(t, x)$, $(t, x) \in [0, \infty) \times \mathbf{R}^1$, be the solution of the Cauchy problem (2.1), (2.2) with random initial condition satisfying Condition A.

Then the finite-dimensional distributions of the process

$$\tilde{X}_t(a) = \frac{t^{1/2 + \alpha/4}}{L^{1/2}(\sqrt{t})} u(t, a\sqrt{t}), \qquad a \in \mathbf{R}^1 \tag{2.5}$$

converge weakly as $t \to \infty$ to the finite-dimensional distributions of the stationary Gaussian process $X(a)$, $a \in \mathbf{R}^1$, with $EX(a) = 0$ and a correlation function of the form

$$R(a, b) = EX(a)\, X(b) = (2\mu)^{-1-\alpha/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1 w_2 \varphi(w_1)\, \varphi(w_2)$$

$$\times \frac{dw_1\, dw_2}{|[w_1 - w_2 - (a-b)/(2\mu)^{1/2}]|^\alpha}$$

$$0 < \alpha < 1, \quad a, b \in \mathbf{R}^1 \tag{2.6}$$

Remark 2.1. If there exists a *spectral density* $f(\lambda)$, $\lambda \in \mathbf{R}^1$, of the Gaussian process

$$\xi(x) = \int_{-\infty}^{\infty} e^{i\lambda x} [f(|\lambda|)]^{1/2}\, W(d\lambda), \qquad x \in \mathbb{R}^1$$

where $W(\cdot)$ is the complex Gaussian white noise and the function $f(\lambda)$ is supposed to be decreasing for $|\lambda| \geqslant \lambda_0 > 0$, then[15] the limiting Gaussian process $X(a)$, $a \in \mathbf{R}^1$, in Theorem 2.1 can be represented in the following way:

$$X(a) = -\frac{1}{i} \left[\frac{2}{\Gamma(\alpha+1) \cos(\alpha\pi/2)} \right]^{1/2} \int_{-\infty}^{\infty} e^{i\lambda a} g(\lambda)\, W(d\lambda) \tag{2.7}$$

where

$$g(\lambda) = \exp(-\mu\lambda^2)\, |\lambda|^{(\alpha-1)/2}\, \lambda, \qquad \lambda \in \mathbf{R}^1$$

Using a Tauberian theorem[12] under Condition A we have the following asymptotic representation:

$$f(\lambda) = f(|\lambda|) \approx \alpha\lambda^{\alpha-1} L\left(\frac{1}{\lambda}\right) \left[2\Gamma\left(1 + \frac{\alpha}{2}\right) 2^\alpha \sqrt{\pi}\, \Gamma^{-1}\left(\frac{1-\alpha}{2}\right) \right]^{-1}$$

$$\lambda \to 0+, \quad 0 < \alpha < 1$$

Using (2.7), we have $g(0) = 0$.

3. MAIN RESULT

Introduce the uniform distance between distribution functions

$$\Delta_t = \sup_{-\infty < z < \infty} \left| P\left\{ \frac{1}{\sigma} \tilde{X}_t(a) \leqslant z \right\} - \Phi(z) \right|, \qquad \sigma^2 = R(a, a) \tag{3.1}$$

where $\tilde{X}_t(a)$, $a \in \mathbf{R}^1$, is defined by (2.5), $\Phi(z)$ is defined by (2.4), and $R(a, b)$ is defined by (2.6).

The main result of this paper describes the rate of convergence to the normal law as $t \to \infty$. This result is presented in the next theorem.

Theorem 3.1. Let $u(t, x)$, $(t, x) \in [0, \infty) \times \mathbf{R}^1$, be the solution of the Cauchy problem (2.1), (2.2) with random initial condition satisfying Condition A for $0 < \alpha < 1/2$. Then the following quantity exists:

$$\overline{\lim_{t \to \infty}} \, [t^{\alpha/6}/[L(\sqrt{t})]^{1/3}] \, \Delta_t$$

and is bounded by

$$\tfrac{3}{2} \nu_1^{2/3} \nu_2^{1/3}$$

Δ_t is defined by (3.1) and

$$\nu_1 = 1 + \frac{1}{(2\pi)^{1/2}}$$

$$\nu_2 = \theta^2 (2\mu)^{2 - \alpha/2} \left\{ 2K \left[e^{1/(2\mu^2)} - \left(1 + \frac{1}{4\mu^2}\right) e^{1/(4\mu^2)} \right] e^{-1/(4\mu^2)} + \frac{M}{8\mu^4} \right\}$$

θ is an arbitrary fixed constant such that $\theta > 1$, and

$$K = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1 w_2 \varphi(w_1) \, \varphi(w_2) \, \frac{dw_1 \, dw_2}{|w_1 - w_2|^{2\alpha}}$$

$$\times \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1 w_2 \varphi(w_1) \, \varphi(w_2) \, \frac{dw_1 \, dw_2}{|w_1 - w_2|^{\alpha}} \right]^{-1}$$

$$M = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(w_1) \, \varphi(w_2) \, \frac{dw_1 \, dw_2}{|w_1 - w_2|^{\alpha}}$$

Before proving Theorem 3.1, we mention some well-known results.

Lemma 3.1 (ref. 20, p. 28). Let X, Y be two arbitrary random variables. Then for any $\varepsilon > 0$

$$\sup_z |\mathbf{P}\{X + Y \leqslant z\} - \Phi(z)| \leqslant \sup_z |\mathbf{P}\{X \leqslant z\} - \Phi(z)| + \frac{\varepsilon}{(2\pi)^{1/2}} + \mathbf{P}\{|Y| > \varepsilon\}$$

where $\Phi(z)$ defined by (2.4).

Lemma 3.2 (ref. 19, Lemma 1). Let X, U be two arbitrary random variables. Then for any $\varepsilon > 0$

$$\sup_z |\mathbf{P}\{X \leqslant zU\} - \Phi(z)| \leqslant \sup_z |\mathbf{P}\{X \leqslant z\} - \Phi(z)| + \mathbf{P}\{|U - 1| > \varepsilon\} + \varepsilon$$

Lemma 3.3. Let X, Y, U be any random variables and $U > 0$. Then for any $\varepsilon > 0$

$$\sup_z |\mathbf{P}\{(X+Y)/U \leqslant z\} - \Phi(z)|$$

$$\leqslant \sup_z |\mathbf{P}\{X \leqslant z\} - \Phi(z)|$$

$$+ \mathbf{P}\{|Y| > \varepsilon\} + \frac{\varepsilon}{(2\pi)^{1/2}} + \mathbf{P}\{|U-1| > \varepsilon\} + \varepsilon$$

Proof. Lemma 3.3 follows from Lemmas 3.1 and 3.2.

Lemma 3.4. Let W, T be two arbitrary random variables. Then for any $\varepsilon > 0$

$$\mathbf{P}\{|W+T| > \varepsilon\} \leqslant \mathbf{P}\{|W| > \varepsilon\delta\} + \mathbf{P}\{|T| > \varepsilon(1-\delta)\}, \qquad 0 < \delta < 1$$

Proof. Obvious.

Let

$$H_m(u) = (-1)^m e^{u^2/2} \frac{d^m}{du^m} e^{-u^2/2}, \qquad u \in \mathbf{R}^1, \quad m = 0, 1, 2,...$$

be the Hermite polynomials with the leading coefficient equal to 1. As it is well known, they form a complete orthogonal system in the Hilbert space $L_2(\mathbf{R}^1, \varphi(u)\,du)$.

Lemma 3.5. Let (ξ, η) be a Gaussian vector with $E\xi = E\eta = 0$, $E\xi^2 = E\eta^2 = 1$, $E\xi\eta = \rho$; then for all $m \geqslant 0$, $q \geqslant 0$

$$EH_m(\xi)\, H_q(\eta) = \delta_{mq} \rho^m m!$$

where δ_{mq} is the usual Kronecker symbol.

The statement of Lemma 3.5 is well known (see, for example, ref. 10, p. 55).

Proof of Theorem 3.1. If $G(u)$ is a real function such that $EG(\xi(0)) < \infty$ in $L_2(\mathbf{R}^1, \varphi(u)\,du)$ we have the following expansion:

$$G(u) = \sum_{k=0}^{\infty} C_k H_k(u)/k!, \quad C_k = \int_{-\infty}^{\infty} G(u)\, H_k(u)\, \varphi(u)\, du, \quad k = 0, 1,... \tag{3.2}$$

By Parseval's equality it follows that

$$\int_{-\infty}^{\infty} G^2(u)\,\varphi(u)\,du = \sum_{k=0}^{\infty} C_k^2/k! < \infty \tag{3.3}$$

In particular, from (3.2) the coefficients of Hermite's expansion of the function $G(u) = \exp(-u/2\mu)$, $u \in \mathbf{R}^1$, are given by

$$\begin{aligned} C_0 &= \exp\left(\frac{1}{8\mu^2}\right), \qquad C_1 = -\frac{1}{2\mu}\exp\left(\frac{1}{8\mu^2}\right) \\ C_k &= \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \exp\left(-\frac{u+u^2\mu}{2\mu}\right) H_k(u)\,du, \qquad k = 2, 3,... \end{aligned} \tag{3.4}$$

All this implies the following expansion in the Hilbert space $L_2(\Omega)$:

$$\exp\left(-\frac{\xi(y)}{2\mu}\right) = \sum_{k=0}^{\infty} C_k \frac{H_k(\xi(y))}{k!} \tag{3.5}$$

where the C_k are defined by (3.4).

We consider the random variables

$$\eta_k(a, t) = \int_{-t}^{t} \frac{a\sqrt{t-y}}{t}\, g(t, a\sqrt{t-y})\, H_k(\xi(y))\,dy, \qquad k = 0, 1,...$$

In order to apply Lemma 3.3, we represent $\tilde{X}_t(a)/\sigma$ using (2.3), (3.5) as

$$\tilde{X}_t(a)/\sigma = (X_t + Y_t)/U_t \tag{3.6}$$

where

$$\begin{aligned} X_t &= e^{-1/(8\mu^2)} A_t C_1 \eta_1(a, t) \\ &= e^{-1/(8\mu^2)} A_t C_1 \int_{-t}^{t} \frac{a\sqrt{t-y}}{t}\, g(t, a\sqrt{t-y})\, H_1(\xi(y))\,dy \\ Y_t &= e^{-1/(8\mu^2)} A_t \left[\sum_{k\geqslant 2} \frac{C_k}{k!}\eta_k(a, t) \right. \\ &\quad \left. + \int_{|y|>t} \frac{a\sqrt{t-y}}{t}\, g(t, a\sqrt{t-y})\, e^{-\xi(y)/2\mu}\,dy\right] \\ &= e^{-1/(8\mu^2)} A_t [W_t + T_t] \\ U_t &= J(a\sqrt{t}, t)\, e^{-1/(8\mu^2)}, \qquad A_t = t^{1/2+\alpha/4}/[\sigma L^{1/2}(\sqrt{t})] \end{aligned}$$

We note that $C_0\eta_0(a, t) \to 0$, $t \to \infty$.

From Lemma 3.5 we have that

$$E\eta_k(a,t)\,\eta_j(a,t)=\delta_k^j \operatorname{Var}\eta_k(a,t), \qquad k\geqslant 1, \quad j\geqslant 1 \tag{3.7}$$

where

$$\begin{aligned}\operatorname{Var}\eta_k(a,t) &= \psi_k^2(t)\\ &= k!\int_{-t}^{t}\int_{-t}^{t}\frac{a\sqrt{t}-y_1}{t}\,\frac{a\sqrt{t}-y_2}{t}\\ &\quad\times g(t,a\sqrt{t}-y_1)\,g(t,a\sqrt{t}-y_2)\,B^k(|y_1-y_2|)\,dy_1\,dy_2\end{aligned}$$

After the transformation

$$\frac{w_i^2}{2}=\frac{(a\sqrt{t}-y_i)^2}{4\mu t}, \qquad i=1,2$$

by using the properties of slowly varying functions (see, for example, ref. 10, p. 56), we have as $0<\alpha<1/2$ and $t\to\infty$

$$\begin{aligned}\psi_k^2(t) &= \frac{2\mu k!}{(2\mu)^{k\alpha/2}\,t^{1+k\alpha/2}}\iint_{w_i\in A(a,t);\, i=1,2} w_1 w_2\varphi(w_1)\,\varphi(w_2)\\ &\quad\times\frac{L^k((2\mu t)^{1/2}\,|w_1-w_2|)}{|w_1-w_2|^{k\alpha}}\,dw_1\,dw_2\\ &= c_1(k,\alpha)\,\frac{2\mu k!}{(2\mu)^{k\alpha/2}}\,\frac{L^k((2\mu t)^{1/2})}{t^{1+k\alpha/2}}\,[1+o(1)]\end{aligned}$$

where

$$A(a,t)=\left[\frac{a}{(2\mu)^{1/2}}-\left(\frac{t}{2\mu}\right)^{1/2},\ \frac{a}{(2\mu)^{1/2}}+\left(\frac{t}{2\mu}\right)^{1/2}\right]$$

$$c_1(k,\alpha)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_1w_2\varphi(w_1)\,\varphi(w_2)\,\frac{dw_1\,dw_2}{|w_1-w_2|^{k\alpha}}, \qquad 0<\alpha<\frac{1}{k}$$

In particular, as $t\to\infty$,

$$\psi_1^2(t)=\operatorname{Var}\eta_1(a,t)=(2\mu)^{1-\alpha/2}\,c_1(1,\alpha)\,\frac{L(\sqrt{t})}{t^{1+\alpha/2}}\,[1+o(1)], \qquad 0<\alpha<1$$

We note that the random variable

$$X_t=e^{-1/(8\mu^2)}A_tC_1\eta_1(a,t)=\frac{\eta_1(a,t)}{[\operatorname{Var}\eta_1(a,t)]^{1/2}}$$

is a standard normal random variable for any $t>0$ in view of Condition A, the expression for C_1 [see (3.4)], and fact that $H_1(u)=u$.

So we have

$$\sup_z |\mathbf{P}\{X_t \leqslant z\} - \Phi(z)| = 0 \tag{3.8}$$

From (3.7) we have that

$$\begin{aligned}\operatorname{Var}[A_t W_t e^{-1/(8\mu^2)}] &= A_t^2 e^{-1/(4\mu^2)} \operatorname{Var}\left[\sum_{k\geqslant 2} \frac{C_k}{k!} \eta_k(a,t)\right] \\ &= A_t^2 e^{-1/(4\mu^2)} \sum_{k=2}^{\infty} \frac{C_k^2}{k!} \operatorname{Var} \eta_k(a,t)\end{aligned}$$

Dividing the integrals in the expression of $\psi_k^2(t)$ into several parts and using elementary inequalities for the estimation of each part, it is easy to see that

$$\frac{\operatorname{Var}\eta_k(a,t)}{k!} \leqslant \frac{\operatorname{Var}\eta_r(a,t)}{r!} \qquad \text{for} \quad r \leqslant k$$

and thus for $0<\alpha<1/2$

$$\begin{aligned}&\operatorname{Var}[A_t W_t e^{-1/(8\mu^2)}] \\ &\quad\leqslant A_t^2 \frac{\psi_2^2(t)}{2} e^{-1/(4\mu^2)} \sum_{k=2}^{\infty} \frac{C_k^2}{k!} \\ &\quad= \frac{t^{1+\alpha/2}}{\sigma^2 L(\sqrt{t})} \frac{2\mu}{(\sqrt{2\mu})^{2\alpha}} \frac{L^2(\sqrt{t})}{t^{1+\alpha}} \\ &\qquad\times\left[\iint_{w_i\in A(a,t);\, i=1,2} w_1 w_2 \varphi(w_1)\, \varphi(w_2)\right. \\ &\qquad\left.\times \frac{L^2((2\mu t)^{1/2}\,|w_1-w_2|)}{L^2(\sqrt{t})\,|w_1-w_2|^{2\alpha}}\, dw_1\, dw_2\right] \\ &\qquad\times e^{-1/(4\mu^2)} \sum_{k=2}^{\infty} \frac{C_k^2}{k!} \\ &\quad= \frac{L(\sqrt{t})}{t^{\alpha/2}} K_t \frac{1}{(\sqrt{2\mu})^{\alpha/2}\, C_1^2} \sum_{k=2}^{\infty} \frac{C_k^2}{k!}\end{aligned} \tag{3.9}$$

where

$$K_t = \left[\iint_{w_i \in A(a,t);\ i=1,2} w_1 w_2 \varphi(w_1)\, \varphi(w_2) \right.$$
$$\left. \times \frac{L^2((2\mu t)^{1/2}\, |w_1 - w_2|)}{L^2(\sqrt{t})\, |w_1 - w_2|^{2\alpha}}\, dw_1\, dw_2 \right]$$
$$\times \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1 w_2 \varphi(w_1)\, \varphi(w_2)\, \frac{dw_1\, dw_2}{|w_1 - w_2|^{\alpha}} \right]^{-1}$$

We note that

$$\lim_{t\to\infty} K_t = \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1 w_2 \varphi(w_1)\, \varphi(w_2)\, \frac{dw_1\, dw_2}{|w_1 - w_2|^{2\alpha}} \right]$$
$$\times \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1 w_2 \varphi(w_1)\, \varphi(w_2)\, \frac{dw_1\, dw_2}{|w_1 - w_2|^{\alpha}} \right]^{-1}$$
$$= \frac{c_1(2, \alpha)}{c_1(1, \alpha)} = K \tag{3.10}$$

and

$$c_2 = \frac{1}{C_1^2} \sum_{k=2}^{\infty} \frac{C_k^2}{k!}$$
$$= \frac{1}{C_1^2} \left\{ \int_{-\infty}^{\infty} \left[\exp\left(-\frac{u}{2\mu} \right) \right]^2 \varphi(u)\, du - C_0^2 - C_1^2 \right\}$$
$$= \left[e^{1/(2\mu^2)} - \left(1 + \frac{1}{4\mu^2} \right) e^{1/(4\mu^2)} \right] \left[\frac{1}{4\mu^2}\, e^{1/(4\mu^2)} \right]^{-1} \tag{3.11}$$

Applying Chebyshev's inequality, we obtain from (3.9), (3.11) for any $\varepsilon > 0$, $0 < \delta < 1$, that

$$\mathbf{P}\{ |A_t W_t e^{-1/(8\mu^2)}| > \varepsilon\delta \} \leqslant \frac{\theta^2}{\varepsilon^2} (2\mu)^{-\alpha/2}\, c_2\, \frac{L(\sqrt{t})}{t^{\alpha/2}}\, K_t,$$
$$\theta = \frac{1}{\delta} > 1 \tag{3.12}$$

Note that

$$\begin{aligned}
&\operatorname{Var}[A_t T_t e^{-1/(8\mu^2)}]\\
&= A_t^2 e^{-1/(4\mu^2)} \iint_{\mathbf{R}^2 \setminus \{|y_i| \leqslant t,\, i=1,2\}} \frac{a\sqrt{t}-y_1}{t}\,\frac{a\sqrt{t}-y_2}{t}\\
&\quad \times g(t, a\sqrt{t}-y_1)\, g(t, a\sqrt{t}-y_2)\\
&\quad \times E\exp\left\{-\frac{1}{2\mu}[\xi(y_1)+\xi(y_2)]\right\} dy_1\, dy_2\\
&\leqslant A_t^2 e^{1/(4\mu^2)} \left[\iint_{w_i \notin A(a,t);\, i=1,2} + 2\iint_{w_1 \notin A(a,t);\, w_2 \in A(a,t)}\right]\\
&\quad \times |w_1 w_2|\, \varphi(w_1)\, \varphi(w_2)\, dw_1\, dw_2\\
&\leqslant \frac{A_t^2}{(2\pi)^{1/2}} \exp\left\{-\left[\frac{a}{(2\mu)^{1/2}} + \left(\frac{t}{2\mu}\right)^{1/2}\right]^2\right\} e^{1/(4\mu^2)}
\end{aligned}$$

and for any $\varepsilon > 0$, $0 < \delta < 1$ $(\theta = 1/\delta)$

$$\begin{aligned}
&\mathbf{P}\{|A_t T_t e^{-1/(8\mu^2)}| > \varepsilon(1-\delta)\}\\
&\leqslant \frac{\exp\{-[a/(2\mu)^{1/2} + (t/2\mu)^{1/2}]^2\}}{\varepsilon^2 (2\pi)^{1/2}} A_t^2 e^{1/(4\mu^2)} \left(\frac{\theta}{\theta-1}\right)^2
\end{aligned} \tag{3.13}$$

Using Lemma 3.4, we obtain from (3.12) and (3.13) that

$$\begin{aligned}
\mathbf{P}\{|Y_t| > \varepsilon\} \leqslant \frac{1}{\varepsilon^2}\Bigg[&\theta^2 (2\mu)^{-\alpha/2} c_2 \frac{L\sqrt{t}}{t^{\alpha/2}} K_t\\
&+ \frac{\exp\{-[a/(2\mu)^{1/2} + (t/2\mu)^{1/2}]^2\}}{(2\pi)^{1/2}} A_t^2 e^{1/(4\mu^2)} \left(\frac{\theta}{\theta-1}\right)^2\Bigg]
\end{aligned} \tag{3.14}$$

For any $\varepsilon > 0$, $0 < \delta < 1$ $(\theta = 1/\delta)$ we get from (3.8), (3.14), (3.6), and Lemma 3.3 the following estimation for Δ_t:

$$\begin{aligned}
\Delta_t \leqslant &\frac{\varepsilon}{(2\pi)^{1/2}} + \varepsilon + \frac{1}{\varepsilon^2}\left[\theta^2 (2\mu)^{-\alpha/2} c_2 \frac{L(\sqrt{t})}{t^{\alpha/2}} K_t + R_t\right]\\
&+ \mathbf{P}\{|U_t - 1| > \varepsilon\}
\end{aligned} \tag{3.15}$$

where

$$R_t = \frac{A_t}{(2\pi)^{1/2}} \exp\left\{-\left[\frac{a}{(2\mu)^{1/2}} + \left(\frac{t}{2\mu}\right)^{1/2}\right]^2\right\} e^{1/(4\mu^2)} \left(\frac{\theta}{\theta-1}\right)^2$$
$$= o\left(\frac{L(\sqrt{t})}{t^{\alpha/2}}\right) \tag{3.16}$$

as $t \to \infty$.

We note that

$$U_t - 1 = e^{-1/(8\mu^2)} \left[\int_{-t}^{t} + \int_{|y|>t}\right] g(t, a\sqrt{t} - y)(e^{-\xi(y)/2\mu} - e^{1/(8\mu^2)})\, dy$$
$$= \Sigma_1(t) + \Sigma_2(t) + \Sigma_3(t)$$

where

$$\Sigma_1(t) = e^{-1/(8\mu^2)} C_1 \int_{-t}^{t} g(t, a\sqrt{t} - y)\, \xi(y)\, dy$$

$$\Sigma_2(t) = e^{-1/(8\mu^2)} \sum_{k=2}^{\infty} \frac{C_k}{k!} \int_{-t}^{t} g(t, a\sqrt{t} - y)\, H_k(\xi(y))\, dy$$

$$\Sigma_3(t) = e^{-1/(8\mu^2)} \int_{|y|>t} g(t, a\sqrt{t} - y)(e^{-\xi(y)/2\mu} - e^{1/(8\mu^2)})\, dy$$

Estimating and analyzing the limiting behavior of the integrals below shows that

$$\operatorname{Var} \Sigma_1(t) \leqslant (2\mu)^{-\alpha/2-2} M_t(\alpha) \frac{L(\sqrt{t})}{t^{\alpha/2}} \tag{3.17}$$

$$\lim_{t\to\infty} M_t(\alpha) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\varphi(w_1)\, \varphi(w_2)}{|w_1 - w_2|^\alpha}\, dw_1\, dw_2 = M$$

$$\operatorname{Var} \Sigma_2(t) \leqslant (2\mu)^{-\alpha} e^{-1/(4\mu^2)} M_t(2\alpha) \frac{L^2(\sqrt{t})}{t^\alpha} c_2 \tag{3.18}$$

$$\operatorname{Var} \Sigma_3(t) \leqslant \frac{c_3}{\pi} \exp\left\{-\frac{1}{2}\left[\frac{a}{(2\mu)^{1/2}} + \left(\frac{t}{2\mu}\right)^{1/2}\right]^2\right\}$$
$$\times \left[\frac{a}{(2\mu)^{1/2}} + \left(\frac{t}{2\mu}\right)^{1/2}\right]^{-1}, \qquad c_3 = \text{const} > 0 \tag{3.19}$$

Applying Lemma 3.4 and Chebyshev's inequality, we obtain that

$$\mathbf{P}\{|U_t-1|>\varepsilon\}$$
$$\leqslant\frac{1}{\varepsilon^2}\left\{\frac{\theta^2 M_t(\alpha)\,L(\sqrt{t})}{t^{\alpha/2}}+c_4 Q_t\right\},\qquad c_4=\text{const}>0 \tag{3.20}$$

where

$$Q_t=o\left(\frac{L(\sqrt{t})}{t^{\alpha/2}}\right) \tag{3.21}$$

as $t\to\infty$.

From (3.15) and (3.20) we have

$$\Delta_t\leqslant\varepsilon\left(1+\frac{1}{(2\pi)^{1/2}}\right)+\frac{1}{\varepsilon^2}\left[\frac{\theta^2 L(\sqrt{t})}{(2\mu)^{\alpha/2}\,t^{\alpha/2}}\left(c_2K_t+\frac{M_t(\alpha)}{4\mu^2}\right)+R_t+c_4Q_t\right]$$

In order to minimize the right-hand side of the last inequality, set

$$\varepsilon=\frac{L^{1/3}(\sqrt{t})}{t^{\alpha/6}}\left(1+\frac{1}{\sqrt{2}}\right)^{-1/3}(2\mu)^{-\alpha/6}\left(2c_2K_t+\frac{M_t(\alpha)}{2\mu^2}\right)^{1/3}$$

Thus we derive the following inequality:

$$\Delta_t\leqslant\frac{L^{1/3}(\sqrt{t})}{t^{\alpha/6}}\left[\nu_1^{2/3}\nu_2^{1/3}+\frac{\nu_1^{2/3}\nu_2^{1/3}}{2}+\frac{t^{\alpha/2}}{L(\sqrt{t})}(R_tc_5+c_6Q_t)\right]$$

where c_5 and c_6 are some positive constants.

From the last relationship and (3.10), (3.16), and (3.21), Theorem 3.1 follows.

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